

A NEW PROGRESS ON WEAK DIRAC CONJECTURE

HOANG HA PHAM AND TIEN CUONG PHI

ABSTRACT. In 2014, Payne-Wood proved that every non-collinear set P of n points in the Euclidean plane contains a point in at least $\frac{n}{37}$ lines determined by P . This is a remarkable answer for the conjecture, which was proposed by Erdős, that every non-collinear set P of n points contains a point in at least $\frac{n}{c_1}$ lines determined by P , for some constant c_1 . In this article, we refine the result of Payne-Wood to give that every non-collinear set P of n points contains a point in at least $\frac{n}{26} + 2$ lines determined by P . Moreover, we also discuss some relations on theorem Beck that every set P of n points with at most l collinear determines at least $\frac{1}{61}n(n-l)$ lines.

1. INTRODUCTION

Let P be a set of points in the Euclidean plane. A line that contains at least two points in P is said to be determined by P .

In 1951, G. Dirac ([4]) made the following conjecture, which remains unsolved:

Conjecture 1 (Strong Dirac Conjecture). Every non-collinear set P of n points in the plane contains a point in at least $\frac{n}{2} - c_0$ of the lines determined by P , for some constant c_0 .

In 2011, J. Akiyama, H. Ito, M. Kobayashi, and G. Nakamura ([2]) gave some examples to show that the $\frac{n}{2}$ bound would be tight. We note that if P is non-collinear and contains $\frac{n}{2}$ or more collinear points, then Dirac's Conjecture holds. Thus we may assume that P does not contain $\frac{n}{2}$ collinear points, and $n \geq 5$.

In 1961, P. Erdős ([5]) proposed the following weakened conjecture.

Conjecture 2 (Weak Dirac Conjecture). Every non-collinear set P of n points contains a point in at least $\frac{n}{c_1}$ lines determined by P , for some constant c_1 .

2010 *Mathematics Subject Classification.* 52C10, 52C30.

Key words and phrases. Arrangement of points, Incident-line-number, Dirac conjecture, lines with few point.

In 1983, Beck ([3]) and Szemerédi-Trotter ([18]) proved the Weak Dirac Conjecture for the case c_1 but it is unspecified or very large. In 2014, Payne-Wood ([15]) proved the following theorem:

Theorem 1. *Every non-collinear set P of n points contains a point in at least $\frac{n}{37}$ lines determined by P .*

For the first purpose of this article, we would like to give a new progress for the Weak Dirac conjecture. In particular, we prove the following:

Main theorem 1. *Every non-collinear set P of n points contains a point in at least $\frac{n}{26} + 2$ lines determined by P .*

Moreover, relate to work on the Weak Dirac Conjecture, Beck gave the number of lines determined by P . He proved the following theorem.

Theorem 2. ([3]) *Every set P of n points with at most l collinear determines at least $c_2 n(n-l)$ lines, for some constant c_2 .*

In 2014, Payne - Wood also gave a remarkable improvement of Beck's theorem by proving the following.

Theorem 3. ([15]) *Every set P of n points with at most l collinear determines at least $\frac{1}{98}n(n-l)$ lines.*

We note that the number 98 can be instead by 93. The details can be found in [14].

For the final purpose, we would like to give some results for the number of lines with few points from n points in plane. Then, we also give the following theorems.

Main theorem 2. *Every set P of n points with at most l collinear determines at least $\frac{1}{61}n(n-l)$ lines.*

Main theorem 3. *Every set P of n points with at most l collinear determines at least $\frac{1}{122}n(n-l)$ lines with at most 3 points.*

2. AUXILIARY RESULTS

We list here some known results which are very helpful for the proofs of the main theorems.

The crossing number of a graph G , denoted by $cr(G)$, is the minimum number of crossings

in a drawing of G . The following version due to J. Pach, R. Radoičić, G. Tardos and G. Tóth [13] is the strongest to date.

Lemma 4. (*Crossing lemma* [13]). *For every graph with n vertices and $m \geq \frac{103}{16}n$ edges, then*

$$cr(G) \geq \frac{1024m^3}{31827n^2}.$$

We set $E(H)$ to be the set of all edges of a graph H . The visibility graph G of a point set P has vertex set P , where $vw \in E(G)$ whenever the line segment vw contains no other point in P (that is, v and w are consecutive on a line determined by P). For $i \geq 2$, an i -line is a line containing exactly i points in P . Let s_i be the number of i -lines. Let G_i be the spanning subgraph of the visibility graph of P consisting of all edges in j -lines where $j \geq i$. Note that since each i -line contributes $i-1$ edges, $|E(G_i)| = \sum_{j \geq i} (j-1)s_j$. We introduce some useful results:

Theorem 5. (*Hirzebruch's Inequality* [10]). *Let P be a set of n points with at most $n-3$ collinear. Then*

$$s_2 + \frac{3}{4}s_3 \geq n + \sum_{i \geq 5} (2i-9)s_i.$$

Theorem 6. (*Szemerédi-Trotter* [18]). *Let α and β be positive constants such that every graph H with n vertices and $m \geq \alpha n$ edges satisfies*

$$cr(H) \geq \frac{m^3}{\beta n^2}$$

Let P be a set of n points in the plane. Then

$$\begin{aligned} a) \quad |E(G_i)| &= \sum_{j \geq i} (j-1)s_j \leq \max\{\alpha n, \frac{\beta n^2}{2(i-1)^2}\}, \\ b) \quad \sum_{j \geq i} s_j &\leq \max\{\frac{\alpha n}{i-1}, \frac{\beta n^2}{2(i-1)^3}\}. \end{aligned}$$

3. A NEW PROGRESS ON WEAK DIRAC'S CONJECTURE

In order to get the main theorem 1, we refine the method of Payne-Wood to find the largest number ε such that every set P of n non-collinear points in the plane at most $\varepsilon n + 2$ collinear points, the arrangement of P has at least $\varepsilon n^2 + 2n$ point-line incidents. We start by the following result.

Theorem 7. *Let α and β be positive constants such that every graph G with n vertices and $m \geq \alpha n$ edges satisfies $cr(G) \geq \frac{m^3}{\beta n^2}$.*

Fix two integers $c \geq 8, 0 \leq q \leq 3$ and a real number $\epsilon \in (0; \frac{1}{2}), \epsilon n \geq 2$. Let $h := \frac{c(c-2)}{5c-18}$. Then for every set P of n points in the plane with at most $\epsilon n + q$ collinear points, the arrangement of P has at least $\delta n^2 + rn$ point-line incident, where

$$\delta = \frac{1}{h+1} \left(1 - \epsilon\alpha - \frac{\beta}{2} \left(\frac{-18(c-2)}{c^3(5c-18)} + \sum_{i \geq c} \frac{i+1}{i^3} \right) \right),$$

$$r = \frac{2h-1+\alpha}{h+1}.$$

Proof. Let $J = \{2; 3; \dots; \lfloor \epsilon n \rfloor + q\}$ and assume that $\epsilon n \geq 2$. Considering the visibility graph G of P and its subgraphs G_i as defined previously. Let k be the minimum integer such that $|E(G_k)| \leq \alpha n$. If there is no such k then let $k = \lfloor \epsilon n \rfloor + q + 1$. An integer $i \in J$ is *large* if $i \geq k$, and is *small* if $i \leq c$. An integer in J that is neither small nor large is *medium*.

Recall that an i -line is a line containing exactly i points in P . An i -pair is a pair of points in an i -line. A *small pair* is an i -pair for some small i . Define *large pair*, *medium pair* analogously. Let P_S, P_M and P_L denote the number of small, medium and large pairs respectively. An i -incidence is an incidence between a point of P and an i -line. A *small incidence* is an i -incidence for some small i , and define *medium*, *large incidences* analogously. Let I_S, I_M and I_L denote the number of small, medium and large incidences respectively and let I denote the total number of incidences. Since every s_i has i points incidence with its, then

$$I = \sum_{i \in J} i s_i = I_S + I_M + I_L$$

Because P has no more than $\frac{n}{2}$ collinear points and $n \geq 5$, thus $\lfloor \epsilon n \rfloor + q \leq \lfloor \frac{n}{2} \rfloor \leq n - 3$. Therefore, for n points of P has no more than $n - 3$ collinear points. Applying the Hirzebruch's Inequality (Theorem 5), we have

$$s_2 + \frac{3}{4}s_3 \geq n + \sum_{i \geq 5} (2i - 9)s_i.$$

Since $h > 0$ then,

$$hs_2 + \frac{3}{4}hs_3 - hn - h \sum_{i \geq 5} (2i - 9)s_i \geq 0.$$

$$\begin{aligned}
P_S &= \sum_{i=2}^c \binom{i}{2} s_i \\
&= s_2 + 3s_3 + 6s_4 + \sum_{i=5}^c \binom{i}{2} s_i \\
&\leq (h+1)s_2 + \left(\frac{3h}{4} + 3\right)s_3 + 6s_4 + \sum_{i=5}^c \binom{i}{2} s_i - hn - h \sum_{i \geq 5} (2i - 9)s_i \\
&= \frac{h+1}{2}.2s_2 + \frac{h+4}{4}.3s_3 + \frac{3}{2}.4s_4 + \sum_{i=5}^c \left(\frac{i-1}{2} - 2h + \frac{9h}{i}\right) is_i \\
&\quad - h \sum_{i=c+1}^{k-1} (2i - 9)s_i - h \sum_{i \geq k} (2i - 9)s_i - hn \\
&\leq \frac{h+1}{2}.2s_2 + \frac{h+4}{4}.3s_3 + \frac{3}{2}.4s_4 + \sum_{i=5}^c \left(\frac{i-1}{2} - 2h + \frac{9h}{i}\right) is_i \\
&\quad - h \sum_{i=c+1}^{k-1} \left(2 - \frac{9}{c+1}\right) is_i - h \sum_{i \geq k} \left(2 - \frac{7}{c}\right)(i-1)s_i - hn.
\end{aligned}$$

Setting $X := \max\left\{\frac{h+1}{2}; \frac{h+4}{4}; \frac{3}{2}; \max_{5 \leq i \leq c} \left(\frac{i-1}{2} - 2h + \frac{9h}{i}\right)\right\}$ implies that,

$$P_S \leq XI_S - h \sum_{i=c+1}^{k-1} \left(2 - \frac{9}{c+1}\right) is_i - h \sum_{i \geq k} \left(2 - \frac{7}{c}\right)(i-1)s_i - hn. \quad (3.1)$$

Let $\gamma(h, i) = \frac{i-1}{2} - 2h + \frac{9h}{i}$ for $5 \leq i \leq c$.

We have: $\gamma_i'' \geq 0 \quad \forall i \in (5, c) \Rightarrow \gamma(h, i)_{\max} = \gamma(h, 5) = 2 - \frac{h}{5}$

or $\gamma(h, i)_{\max} = \gamma(h, c) = \frac{c-1}{2} - 2h + \frac{9h}{c}$ for $c \geq 8$.

Clearly, $h(c) = \frac{c(c-2)}{5c-18}$ has minimum value $\frac{24}{11}$ when $c = 8$. Hence,

$$\begin{aligned}\frac{h+1}{2} &\geq \frac{3}{2} \\ \frac{h+1}{2} &\geq \frac{h+4}{4} \\ \frac{h+1}{2} &\geq 2 - \frac{h}{5} \\ \frac{h+1}{2} &= \frac{c-1}{2} - 2h + \frac{9h}{c}.\end{aligned}$$

Thus, $X = \frac{h+1}{2}$.

On the other hand, if $i \in J$ is medium ($c < i < k$) then i is not large. Therefore, $\sum_{j \geq i} (j-1)s_j > \alpha n$. Because if $\sum_{j \geq i} (j-1)s_j \leq \alpha n$ then $|E(G_i)| \leq \alpha n$, contradict with minimum property of k . Using part (a) and (b) of the Szemerdi- Trotter theorem 6,

$$\sum_{j \geq i} js_j = \sum_{j \geq i} (j-1)s_j + \sum_{j \geq i} s_j \leq \frac{\beta n^2}{2(i-1)^2} + \frac{\beta n^2}{2(i-1)^3} = \frac{\beta n^2 i}{2(i-1)^3}. \quad (3.2)$$

Given X as above, we have

$$\begin{aligned}P_M - XI_M &= \left(\sum_{i=c+1}^{k-1} \binom{i}{2} s_i \right) - X \left(\sum_{i=c+1}^{k-1} i s_i \right) \\ &= \frac{1}{2} \sum_{i=c+1}^{k-1} (i-1-2X) i s_i.\end{aligned}$$

Combining with (3.1), we get

$$P_S + P_M \leq XI_S - hn + XI_M + \frac{1}{2} \sum_{i=c+1}^{k-1} \left(i-1-2X-4h + \frac{18h}{c+1} \right) i s_i - h \left(2 - \frac{7}{c} \right) |E(G_k)|. \quad (3.3)$$

We define

$$\begin{aligned}Y &= c - 5h - 2 + \frac{18h}{c+1} \\ &= c - 2 - 5 \frac{c(c-2)}{5c-18} + \frac{18c(c-2)}{(c+1)(5c-18)} \\ &= \frac{-18(c-2)}{(c+1)(5c-18)}.\end{aligned}$$

This implies $-1 < Y < 0$ with $c \geq 8$. Thus we have,

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=c+1}^{k-1} \left(i - 1 - 2X - 4h + \frac{18h}{c+1} \right) i s_i = \frac{1}{2} \sum_{i=c+1}^{k-1} (i - c + Y) i s_i \\ &= \frac{1}{2} \left(\sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} j s_j \right) + \frac{Y}{2} \left(\sum_{i=c+1}^{k-1} i s_i \right) \\ &\leq \frac{1}{2} \left(\sum_{i=c+1}^{k-1} \sum_{j \geq i}^{k-1} j s_j \right) + \frac{Y}{2} \left(\sum_{i \geq c+1} i s_i \right). \end{aligned}$$

Applying (3.2) and $Y + 1 > 0$, this yields

$$T \leq \frac{1}{2} \sum_{i \geq c+1} \frac{\beta n^2 i}{2(i-1)^3} + \frac{Y}{2} \cdot \frac{\beta n^2 (c+1)}{2c^3} = \frac{\beta n^2}{4} \left(Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right). \quad (3.4)$$

Finally, we have

$$P_L - X I_L = \sum_{i=k}^{\lfloor \varepsilon n \rfloor + q} \binom{i}{2} s_i - X \sum_{i \geq k} i s_i \leq \frac{\varepsilon n + q}{2} \sum_{i \geq k} (i-1) s_i - X \sum_{i \geq k} (i-1) s_i = \left(\frac{\varepsilon n + q}{2} - X \right) |E(G_k)|. \quad (3.5)$$

Combining (3.3), (3.4), (3.5), we get

$$\begin{aligned} P_S + P_M + P_L &\leq X(I_S + I_M + I_L) - hn \\ &\quad + \frac{\beta n^2}{4} \left(Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{1}{2} (\varepsilon n + q - 2X - 4h + \frac{7h}{c}) |E(G_k)| \\ &\leq XI - hn + \frac{\beta n^2}{4} \left(Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{1}{2} (\varepsilon n - 2) |E(G_k)| \quad (\text{by } 1 \leq q \leq 3, c \geq 8) \\ &\leq XI - hn + \frac{\beta n^2}{4} \left(Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{1}{2} (\varepsilon n - 2) \alpha n. \end{aligned}$$

On the other hand, we have $P_S + P_M + P_L = \binom{n}{2} = \frac{1}{2}(n^2 - n)$.

Thus, we get

$$\begin{aligned} \frac{1}{2}(n^2 - n) &\leq XI - hn + \frac{\beta n^2}{4} \left(Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) + \frac{\varepsilon \alpha n^2}{2} - \alpha n. \\ \Rightarrow I &\geq \frac{1}{2X} \left(1 - \varepsilon \alpha - \frac{\beta}{2} \left(Y \frac{c+1}{c^3} + \sum_{i \geq c} \frac{i+1}{i^3} \right) \right) n^2 + \frac{2h - 1 + \alpha}{2X} n. \end{aligned}$$

Since $X = \frac{h+1}{2}$ and $Y = \frac{-18(c-2)}{c^3(5c-18)}$ then,

$$\begin{aligned} I &\geq \frac{1}{h+1} \left(1 - \epsilon\alpha - \frac{\beta}{2} \left(\frac{-18(c-2)}{c^3(5c-18)} + \sum_{i \geq c} \frac{i+1}{i^3} \right) \right) n^2 + \frac{2h-1+\alpha}{h+1} n \\ &= \delta n^2 + rn. \end{aligned}$$

□

Theorem 8. *Every set P of n non-collinear points in the plane with at most $\frac{n}{26} + 2$ collinear points, the arrangement of P has at least $\frac{n^2}{26} + 2n$ point-line incidents.*

Proof. Case 1. If $0 < \frac{n}{26} < 1$, then the arrangement of P is $n^2 - n > \frac{n^2}{26} + 2n$ by $n \geq 5$.

Case 2. If $1 \leq \frac{n}{26} < 2$, then $I = 2s_2 + 3s_3 \geq s_2 + 3s_3 = \frac{n^2 - n}{2} > \frac{n^2}{26} + 2n$ by $n \geq 26$.

Case 3. If $\frac{n}{26} \geq 2$, then the assumptions of Theorem 7 satisfy with $\epsilon = \frac{1}{26}$, $c = 46$, $q = 2$. We have

$$I \geq \delta n^2 + rn \geq \frac{n^2}{26} + 2n.$$

The proof of Theorem 8 is completed. □

So we now can give the proof of Main theorem 1.

Proof. Let P be a set of n non-collinear points in the plane. If P contains at least $\frac{n}{26} + 2$ collinear points, then every other point is in at least $\frac{n}{26} + 2$ lines P (one through each of the collinear points). Otherwise, by Theorem 8, the arrangement of P has at least $\frac{n^2}{26} + 2n$ incidences, and so some point is incident with at least $\frac{n}{26} + 2$ lines determined by P . Main theorem 1 is proved. □

We note that the number $\varepsilon = \frac{1}{26}$ is best possible in this technic. Indeed, for our purpose, we need $\delta \geq \varepsilon$ to get a constant ε in Theorem 7. Using equivalent transformation,

$$\varepsilon \leq \frac{1 - \frac{\beta}{2} \left(\frac{-18(c-2)}{c^3(5c-18)} + \sum_{i \geq c} \frac{i+1}{i^3} \right)}{h+1+\alpha} = \frac{1 - \frac{\beta}{2} \left(\frac{-18(c-2)}{c^3(5c-18)} + \sum_{i \geq c} \frac{i+1}{i^3} \right)}{\frac{c(c-2)}{5c-18} + 1 + \alpha}.$$

In order to having maximum value ε we need to optimal value c . We define

$$f(c) = \frac{1 - \frac{\beta}{2} \left(\frac{-18(c-2)}{c^3(5c-18)} + \sum_{i \geq c} \frac{i+1}{i^3} \right)}{\frac{c(c-2)}{5c-18} + 1 + \alpha}$$

, for defined constant α, β in Crossing lemma 4. Using Maple application we have that the maximum value of $f(c)$ is at $c = 46$. Hence, we can choose $\varepsilon > \frac{1}{26}$. This shows that $\frac{1}{26}$ is the best constant.

4. THE LINES WITH FEW POINTS

Theorem 9. *Let α, β be positive constants such that every graph H with n vertices and $m \geq \alpha n$ edges satisfies*

$$cr(H) \geq \frac{m^3}{\beta n^2}.$$

Fix an integer $c \geq 29$. Then for every set P of n points in the plane with at most l collinear points, the arrangement of P has at least

$$\left(1 - \frac{\beta}{2} \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \right) \frac{2c-8}{c^2+3c-18} n^2 - \frac{(2c-8)\alpha}{c^2+3c-18} ln.$$

lines with at most c points.

Proof. We define the small, medium and large pairs and lines respectively as in the proof of Theorem 7. Set $h = \frac{c^2 - c - 2}{4c - 16}$, where $c \geq 29$. Thus, $h > 0$. Using the Inequality of Hirzebruch (Theorem 5), we have

$$s_2 + \frac{3h}{4}s_3 - hn - h \sum_{i \geq 5} (2i-9)s_i \geq 0.$$

Now we have,

$$\begin{aligned}
P_S &= \sum_2^c \binom{i}{2} s_i \\
&= s_2 + 3s_3 + 6s_4 + \sum_{i=5}^c \binom{i}{2} s_i \\
&\leq (h+1)s_2 + \left(\frac{3h}{4} + 3\right)s_3 + 6s_4 + \sum_{i=5}^c \binom{i}{2} s_i - hn - h \sum_{i \geq 5} (2i-9)s_i \\
&\leq (h+1)s_2 + \frac{3}{4}(h+4)s_3 + 6s_4 + \sum_{i=5}^c \left(\frac{i(i-1)}{2} - h(2i-9)\right) s_i - hn - h \sum_{i \geq c+1} (2i-9)s_i.
\end{aligned}$$

By $c \geq 29$, it is easy to see that

$$X := h+1 = \max\{h+1; \frac{3}{4}(h+4); 6; \max_{5 \leq i \leq c}(\frac{i(i-1)}{2} - h(2i-9))\},$$

and thus we get

$$P_S \leq XL_S - hn - h \sum_{i \geq c+1} (2i-9)s_i.$$

For the medium index i , we use the Crossing Lemma 4 and part (a) of Theorem 6 to imply that

$$\sum_{j \geq i} (j-1)s_j \leq \frac{\beta n^2}{2(i-1)^2},$$

thus we have

$$\begin{aligned}
P_S + P_M - XL_S &\leq -hn - h \sum_{i \geq c+1} (2i-9)s_i + \sum_{i=c+1}^{k-1} \binom{i}{2} s_i \\
&= -hn - h \sum_{i \geq k} (2i-9)s_i + \sum_{i=c+1}^{k-1} \left(\frac{i(i-1)}{2} - h(2i-9) \right) s_i \\
&= -hn - h \sum_{i \geq k} (2i-9)s_i + \frac{1}{2} \left(\sum_{i=c+1}^{k-1} \left(c - \frac{4hi-18h}{i-1} \right) (i-1)s_i + \sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} (j-1)s_j \right) \\
&= -hn - h \sum_{i \geq k} (2i-9)s_i + \frac{1}{2} \left(\sum_{i=c+1}^{k-1} \left(c - 4h + \frac{14h}{i-1} \right) (i-1)s_i + \sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} (j-1)s_j \right) \\
&\leq -hn - h \sum_{i \geq k} (2i-9)s_i + \frac{1}{2} \left(\sum_{i=c+1}^{k-1} \left(c - 4h + \frac{14h}{c} \right) (i-1)s_i + \sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} (j-1)s_j \right) \\
&= -hn - h \sum_{i \geq k} (2i-9)s_i + \frac{1}{2} \left(\left(c - 4h + \frac{14h}{c} \right) \sum_{i=c+1}^{k-1} (i-1)s_i + \sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} (j-1)s_j \right)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
c - 4h + \frac{14h}{c} &= c - 4 \frac{c^2 - c - 2}{4c - 16} + \frac{7c^2 - 7c - 14}{c(2c - 8)} = \frac{c^2 - 3c - 14}{2c(c - 4)} \\
&\Rightarrow c - 4h + \frac{14h}{c} > 0 \text{ (by } c \geq 29 \text{)}.
\end{aligned}$$

So we get

$$\begin{aligned}
P_S + P_M - XL_S &\leq -hn - h \sum_{i \geq k} (2i-9)s_i + \frac{1}{2} \left(\frac{c^2 - 3c - 14}{2c(c - 4)} \sum_{i \geq c+1} (i-1)s_i + \sum_{j=c+1}^{k-1} \sum_{i \geq j} (i-1)s_i \right) \\
&\leq -hn - h \sum_{i \geq k} (2i-9)s_i + \left(\frac{c^2 - 3c - 14}{2c^3(c - 4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \frac{\beta n^2}{4}.
\end{aligned}$$

Thus, we now have

$$\begin{aligned}
\binom{n}{2} - XL_S &= P_S + P_M + P_L - XL_S \\
&\leq -hn - h \sum_{i \geq k} (2i - 9)s_i + \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \sum_{i=k}^l \binom{i}{2} s_i \\
&= -hn + \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \sum_{i \geq k} \left(\frac{i(i-1)}{2} - 2hi + 9h \right) s_i \\
&= -hn + \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \sum_{i \geq k} \left(\frac{i}{2} - 2h + \frac{7h}{i-1} \right) (i-1)s_i \\
&\leq -hn + \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \frac{l}{2} |E(G_k)| \\
&\leq -hn + \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \frac{\beta n^2}{4} + \frac{l}{2} \alpha n.
\end{aligned}$$

So we get

$$\begin{aligned}
L_S &\geq \left(\frac{1}{2} - \frac{\beta}{4} \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \right) \frac{n^2}{X} + \left(h - \frac{1}{2} - \frac{l\alpha}{2} \right) \frac{n}{X} \\
&\geq \left(\frac{1}{2} - \frac{\beta}{4} \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \right) \frac{n^2}{X} - \frac{l\alpha n}{2X}.
\end{aligned}$$

On the other hand, $X = h + 1 = \frac{c^2 + 3c - 18}{4c - 16}$, we thus get

$$L_S \geq \left(1 - \frac{\beta}{2} \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \right) \frac{2c - 8}{c^2 + 3c - 18} n^2 - \frac{(2c - 8)\alpha}{c^2 + 3c - 18} ln.$$

Theorem 9 is proved. \square

For the case $c = 36$, we get the following.

Corollary 10. *Every set P of n points with at most l collinear determines at least $\frac{1}{39}n^2 - \frac{1}{3}ln$ lines with at most 36 points.*

We now apply Theorem 9 to give the proof of Main theorem 2.

Proof. We may assume that l is the size of the longest line. For some integer $c \geq 29$, then by Theorem 9 we have $L \geq L_S \geq A(c)n^2 - B(c)nl$ for some $A(c)$ and $B(c)$ evident in the theorem. Observe that,

$$\begin{aligned} A(c) &= \left(1 - \frac{\beta}{2} \left(\frac{c^2 - 3c - 14}{2c^3(c-4)} + \sum_{i \geq c} \frac{1}{i^2} \right) \right) \frac{2c-8}{c^2+3c-18} \\ B(c) &= \frac{(2c-8)\alpha}{c^2+3c-18}. \end{aligned}$$

We note that,

$$\begin{aligned} \frac{2A}{1+2B} &\geq \epsilon \\ \Rightarrow A &\geq \frac{\epsilon}{2} + B\epsilon - \frac{\epsilon^2}{2} \\ \Rightarrow An &\geq \frac{\epsilon n}{2} + (B - \frac{\epsilon}{2})\epsilon n \\ \Rightarrow An &\geq \frac{\epsilon n}{2} + (B - \frac{\epsilon}{2})l \\ \Rightarrow An^2 - Bnl &\geq \frac{\epsilon n(n-l)}{2}. \end{aligned}$$

So we can find the maximum of $\frac{2A(c)}{1+2B(c)}$ to get a largest number ϵ . Now, set $c = 44$ we get $\epsilon \leq \frac{1}{30.2}$. So we choose $\epsilon = \frac{1}{30.5}$ to complete Main theorem 2. \square

We now get Main theorem 3 by using Main theorem 2 and the following observation:

Theorem 11. ([15]) *Let P be a set of n non-collinear points in a plane. Then at least half the lines determined by P contain at most 3 points.*

REFERENCES

- [1] Martin Aigner and Günter M. Ziegler, *Proofs from The Book*. Springer, 3rd edn., 2004.
- [2] Jin Akiyama, Hiro Ito, Midori Kobayashi, and Gisaku Nakamura, *Arrangements of n points whose incident-line-numbers are at most $n/2$* . Graphs Combin., 27(3):321-326, 2011.
- [3] József Beck, *On the lattice property of the plane and some problems of Dirac, Motzkin and Erdos in combinatorial geometry*, Combinatorica, 3(3-4):281-297, 1983.
- [4] Gabriel A. Dirac, *Collinearity properties of sets of points*. Quart. J. Math., Oxford Ser. (2), 2:221-227, 1951.
- [5] Paul Erdos, *Some unsolved problems*. Magyar Tud. Akad. Mat. Kutató Int. Közl., 6:221-254, 1961. http://www.renyi.hu/~p_erdos/1961-22.pdf.

- [6] Paul Erdős and George Purdy, *Some combinatorial problems in the plane*. J. Combin.Theory Ser. A, 25(2):205-210, 1978.
- [7] Paul Erdős and George Purdy, *Two combinatorial problems in the plane*. Discrete Comput. Geom., 13(3-4):441-443, 1995.
- [8] Paul Erdős and Endre Szemerédi, *On sums and products of integers*. In Studies in pure mathematics, pp. 213-218. Birkhauser, Basel, 1983.
- [9] Ben J. Green and Terence Tao, *On sets defining few ordinary lines*, Discrete Comput Geom., 50:409-468, 2013.
- [10] Friedrich Hirzebruch, *Singularities of algebraic surfaces and characteristic numbers*. In The Lefschetz Centennial Conference, Part I, vol. 58 of Contemp. Math., pp. 141-155. Amer. Math. Soc., 1986.
- [11] Leroy M. Kelly and William O. J. Moser, *On the number of ordinary lines determined by n points*. Canad. J. Math., 10:210-219, 1958.
- [12] Eberhard Melchior, *Über Vielseite der projektiven Ebene*. Deutsche Math., 5:461-475, 1941.
- [13] János Pach, Radoš Radoičić, Gábor Tardos and Géza Tóth, *Improving the cross-ing lemma by finding more crossings in sparse graphs*. Discrete Comput. Geom., 36(4):527-552, 2006.
- [14] Michael S. Payne, *Combinatorial geometry of point sets with collinearities*. PhD thesis, The University of Melbourne, Department of Mathematics and Statistics, 2014. <http://www.ms.unimelb.edu.au/~mspayne/MichaelPayneThesis.pdf>.
- [15] Michael S. Payne and David R. Wood, *Progress on Dirac's Conjecture*, The electronic journal of combinatorics, 21(2) 2-12, 2014.
- [16] George Purdy, *A proof of a consequence of Diracs conjecture*. Geom. Dedicata, 10(1-4):317-321, 1981.
- [17] László A. Székely, *Crossing numbers and hard Erdos problems in discrete geometry*. Combin. Probab. Comput. 6(3):pp 353-358, 1997.
- [18] Endre Szemerédi and William T. Trotter, Jr, *Extremal problems in discrete geometry*. Combinatorica, 3(3-4):381-392, 1983.

DEPARTMENT OF MATHEMATICS, HANOI NATIONAL UNIVERSITY OF EDUCATION, 136 XUAN THUY STR., HANOI, VIETNAM

E-mail address: ha.ph@hnue.edu.vn; cuong.tienphi@gmail.com